The Tableau WorkBench

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Outline

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TWB At First Glance

- It is generic and extensible
- Modular framework to build theorem provers
- It is based on a general tableau algorithm
- Front-end for tableau calculi

Simple things should be simple, difficult things should be possible.

State Of Art

- Direct Provers
  - Fact/Fact++ (Description logics)
  - LWB (Modal logics, intuitionistic logics)
  - KSAT (modal logic)
  - Lotrec (arbitrary logics)
- Translational Provers
  - SPASS / MSPASS (first order logics, modal logics)
  - Vampire (first order logic)

Target Audience

- Logicians without programming/technical background:
  - to experiment with new logics
  - to prototype theorem provers
  - to compare different technique using the same framework
- Automated reasoning teachers:
  - syntax similar to textbooks
  - no need to learn a programming language (for simple tasks)
- Expert automated reasoner with programming skills:
  - can be used as an embedded prover
  - the user level can be bypassed and each prover can be highly optimized

Outline

Tableau Methods

- Traditional tableau methods:
  - Modular
  - Efficient
  - Simple
- Extensions to the traditional formulation:
  - Histories (termination)
  - Variables
  - Tactic language

Traditional tableau calculi specify what it can be done, not how to do it.

Outline

Definitions

A tableau rule **(name)** $\dfrac{n}{d_1 \ldots d_n}$ (side conditions) is defined as:

- numerator, denominators
- principal formula, side formula
- each $n, d_1 \ldots d_n$ is a list of formula schema, e.g. $A \& B$, where A and B are meta variables
- side conditions

A node in a tableau is composed of a set of formula plus other data structures (Histories and Variables)

- A node is partitioned according to a rule numerator
- A rule is applicable to a node if there exists a partition of the node and all side conditions are true.

Removing Non-Determinism

- Node Choice: Out of many, we select a node to expand (the current node)
- Rule Choice: Given a node, we select a rule that is applicable to the current node
- Formula Choice: Given a node and a rule, we select a partition of the node according to the rule numerator

Backtracking

	Universal	Existential	Linear
No Commit	-	-	(K)
Commit	$(\vee)$	$(K - rec)$	$(\wedge)$

Thus, the double bar "||" intuitively means: "If the sub-tableau for the first i denominators are all open, then explore the $(i+1)^{st}$ sub-tableau . This is opposite to the single bar "|" traditionally used to specify the $(\vee)$-rule : "if the first branch is closed, then explore the second branch".

How big ?

Core library by lines of code (loc)

component	loc
core	464
tableau library	818
syntax library	1624
data-type library	664
application (cli)	226
total	3800

How small ?

Logic modules by lines of code (loc)

logic	loc (tableau)	loc (functions)	total
CPL	18	98	126
K	21	72	93
KT	31	72	103
S4	41	72	113
PLTL	148	194	242

Outline

Tableau calculus for K

$$(\bot)\ \frac{\varphi; \neg\varphi}{\bot} \qquad (\wedge)\ \frac{\varphi \wedge \psi}{\varphi; \psi} \qquad (\vee)\ \frac{\varphi \vee \psi}{\varphi \mid \psi}$$

$$(K)\ \frac{\Diamond\varphi\ ;\ \Box\Sigma\ ;\ \Lambda}{\varphi; \Sigma}$$

Tableau calculus for S4 with histories

$$(\bot)\ \frac{\varphi; \neg\varphi \ :: \ \ldots}{\bot} \qquad (\wedge)\ \frac{\varphi \wedge \psi \ :: \ \ldots}{\varphi; \psi \ :: \ \ldots} \qquad (\vee)\ \frac{\varphi \vee \psi \ :: \ \ldots}{\varphi \ :: \ \ldots \mid \psi \ :: \ \ldots}$$

$$(T)\ \frac{\Box\varphi \ :: \ \Pi, \Sigma}{\varphi \ :: \ \emptyset \ , \ \{\varphi\} \cup \Sigma}\ \varphi \notin \Sigma$$

$$(S4)\ \frac{\Diamond\varphi; \Lambda \ :: \ \Pi, \Sigma}{\varphi; \Sigma \ :: \ \{\Diamond\varphi\} \cup \Pi \ , \ \Sigma}\ \Diamond\varphi \notin \Pi$$

Rule Syntax (1/4)

```
CONNECTIVES
  And,   "_&_",   Two;
  Or,    "_v_",   Two;
  Imp,   "_->_",  One;
  DImp,  "_<->_",  One;
  Not,   "~_",    Zero;
  Dia,   "Dia_",  Zero;
  Box,   "Box_",  Zero
END

HISTORIES
  (PI  : Set of Formula := new Set.set);
  (SIGMA     : Set of Formula := new Set.set)
END
```

Rule Syntax (2/4)

```
RULE Id
{ a } ; { ~ a }
===============
  Close
END
```

$$(\bot)\ \frac{\varphi; \neg\varphi}{\bot}$$

```
RULE And
{ a & b }
==========
  a ; b
END
```

$$(\wedge)\ \frac{\varphi \wedge \psi}{\varphi; \psi}$$

Rule Syntax (3/4)

```
RULE Or
{ a v b }
==========
  a | b
END
```

$$(\vee)\ \frac{\varphi \vee \psi}{\varphi \mid \psi}$$

```
RULE K
{ Dia a } ; Box x ; z
---------------------
        a ; x
END
```

$$(K)\ \frac{\Diamond\varphi \;;\; \Box\Sigma \;;\; \Lambda}{\varphi ; \Sigma}$$

Rule Syntax (4/4)

```
RULE S4
{ Dia a } ; z
----------------------
a ; SIGMA

COND notin(Dia a, PI)
ACTION [ PI := add(Dia a,PI) ]
END (cache)
```

$$(S4)\ \frac{\Diamond\varphi;\Lambda\ ::\ \Pi,\Sigma}{\varphi;\Sigma\ ::\ \{\Diamond\varphi\}\cup\Pi\ ,\ \Sigma}\ \Diamond\varphi\notin\Pi$$

Strategy

- $t = Skip$: t always succeeds
- $t = Fail$: t always fails
- $t = Rule(r)$: t succeeds when rule r is applicable.
- $t = t_1; t_2$: t fails if either t_1 or t_2 fail, succeeds otherwise.
- $t = t_1 | t_2$: t fails if both t_1 and t_2 fail, succeeds otherwise.
- $Repeat(t) = mu(X).(t; X | Skip)$: If tactic t succeeds, the tactic $Repeat(t)$ behaves like $t; Repeat(t)$. If t fails, then the tactic $Repeat(t)$ behaves like $Skip$. Repeat always succeeds.

Outline

Optimizations : Simplification

```
RULE And
{A & B} ; X
====================
A[B]; B[A] ; X[A][B]
END

RULE Or
{ A v B } ; X
===============================
   A ; X[A] | B ; X[B]
END
```

Optimizations : Simplification

```
let rec simpl phi a =
  let rec aux phi a = match a with
    |term (~ b) when b = a -> term(Falsum)
    |term (~ b)    -> term ( ~ [aux phi b] )
    |term (b & c) ->
             term ( [aux phi b] & [aux phi c] )
    |term (b v c) ->
             term ( [aux phi b] v [aux phi c] )
    |_ when phi = a -> term(Verum)
    |_ when phi = (nnf (term ( ~ a ))) ->
             term(Falsum)
    |_ -> a
  in boolean (aux phi a)
```

Optimizations : Simplification

$$\neg\varphi \vee \varphi \longrightarrow \top \quad \varphi \vee \top \longrightarrow \top \quad \varphi \vee \bot \longrightarrow \varphi \quad \varphi \vee \varphi \longrightarrow \varphi$$

$$\neg\varphi \wedge \varphi \longrightarrow \bot \quad \varphi \wedge \top \longrightarrow \varphi \quad \varphi \wedge \bot \longrightarrow \bot \quad \varphi \wedge \varphi \longrightarrow \varphi$$

$$\neg\bot \longrightarrow \top \qquad \neg\top \longrightarrow \bot$$

Benchmarks

k is the naive tableau prover for *k*. The only optimization is caching;

k − nocache is as *k* but without using caching;

ks is as *s*4 but using caching and simplification and semantic branching;

kbj is as *s*4 but using caching and back-jumping and semantic branching;

ksbj uses all the previously mentioned optimization.

Benchmarks

class	twb	lwb
k_branch_p	6	15
k_d4_p	**11**	10
k_dum_p	**18**	16
k_grz_p	**21**	17
k_lin_p	**21**	14
k_path_p	**16**	14
k_ph_p	4	7
k_poly_p	**19**	11
k_t4p_p	**21**	10

class	twb	lwb
k_branch_n	4	11
k_d4_n	**17**	6
k_dum_n	**19**	21
k_grz_n	**21**	21
k_lin_n	**21**	5
k_path_n	**14**	12
k_ph_n	**6**	7
k_poly_n	**20**	21
k_t4p_n	**21**	8

Table: Benchmarks comparing the TWB and LWB for modal logic *K*

Outline

Unified Branching Logic (UB)

- Introduced by Ben-Ari et al. in 1982
- Subsumed by CTL (Emerson et al. 1982)
- It extends Linear time logic without considering the operator Until.

Syntax

$$
\begin{array}{lcl}
p & ::= & p_1 \mid p_2 \mid \cdots \\
\varphi & ::= & p \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \bot \mid \top \\
 & & \mid AG\varphi \mid AF\varphi \mid AX\varphi \\
 & & \mid EG\varphi \mid EF\varphi \mid EX\varphi
\end{array}
$$

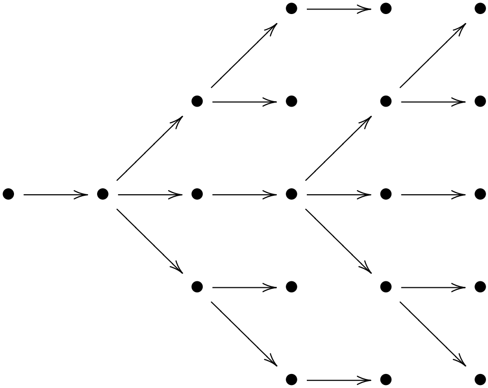
Semantics

Let $M = (S, R, L)$ be a *UB*-model. The truth value $M, s \Vdash \varphi$ of a formula φ in a state $s \in S$ is recursively defined as follows:

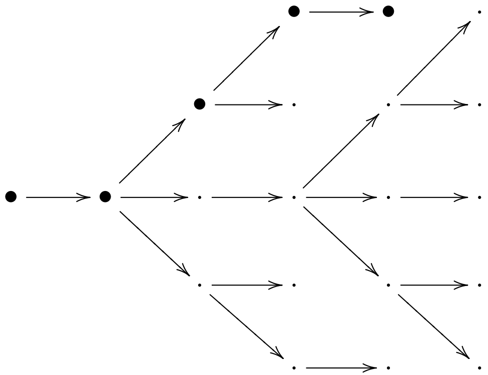
1. $M, s \Vdash p$ iff $p \in L(s)$
2. $M, s \Vdash \neg\varphi$ iff $M, s \nVdash \varphi$
3. $M, s \Vdash \varphi_1 \vee \varphi_2$ iff $M, s \Vdash \varphi_1$ or $M, s \Vdash \varphi_2$
4. $M, s \Vdash \varphi_1 \wedge \varphi_2$ iff $M, s \Vdash \varphi_1$ and $M, s \Vdash \varphi_2$
5. $M, s \Vdash EX\varphi$ iff $\exists t \in S$ such that sRt and $M, t \Vdash \varphi$
6. $M, s \Vdash AF\varphi$ iff $\forall$ fullpaths b with $b_0 = s$, $\exists t \in b$ such that $M, t \Vdash \varphi$
7. $M, s \Vdash EF\varphi$ iff $\exists$ a fullpath b with $b_0 = s$ and $\exists t \in b$ such that $M, t \Vdash \varphi$

The semantic of *AX*, *AG* and *EG* is given by duality with *EX*, *EF* and *AF*, respectively.

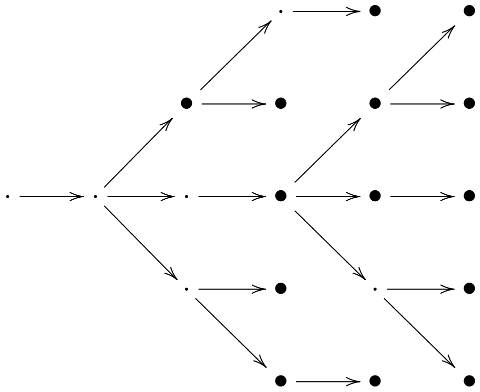
Intuition for AG



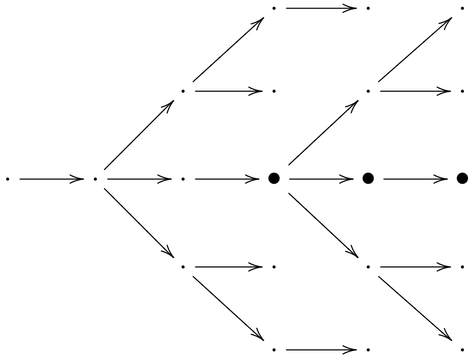
Intuition for EG



Intuition for AF



Intuition for EF



Linear Rules

$$(\wedge)\ \frac{\varphi \wedge \psi\ ;\ \Gamma :: \cdots}{\varphi\ ;\ \psi\ ;\ \Gamma :: \cdots} \qquad (D)\ \frac{\Gamma :: \cdots}{EX\top\ ;\ \Gamma :: \cdots}\ EX\varphi \notin \Gamma$$

$$(AG)\ \frac{AG\varphi\ ;\ \Gamma :: \cdots}{\varphi\ ;\ AXAG\varphi\ ;\ \Gamma :: \cdots} \qquad (EG)\ \frac{EG\varphi\ ;\ \Gamma :: \cdots}{\varphi\ ;\ EXEG\varphi\ ;\ \Gamma :: \cdots}$$

- The (D) enforces seriality
- The (AG) captures the axioms $AG\varphi \leftrightarrow \varphi \wedge AXAG\varphi$
- The (EG) captures the axioms $EG\varphi \leftrightarrow \varphi \wedge EXEG\varphi$

Universally Branching Rules

$$(EF)\ \frac{EF\varphi\ ;\ \Gamma :: Fev, Br :: uev}{\varphi\ ;\ \Gamma :: \{EF\varphi\} \cup Fev, Br :: uev_1 \mid EXEF\varphi\ ;\ \Gamma :: Fev, Br :: uev_2}$$

$$(AF)\ \frac{AF\varphi\ ;\ \Gamma :: Fev, Br :: uev}{\varphi\ ;\ \Gamma :: \{AF\varphi\} \cup Fev, Br :: uev_1 \mid AXAF\varphi\ ;\ \Gamma :: Fev, Br :: uev_2}$$

$$(\vee)\ \frac{\varphi \vee \psi\ ;\ \Gamma :: \cdots :: uev}{\varphi\ ;\ \Gamma :: \cdots :: uev_1 \mid \psi\ ;\ \Gamma :: \cdots :: uev_2}$$

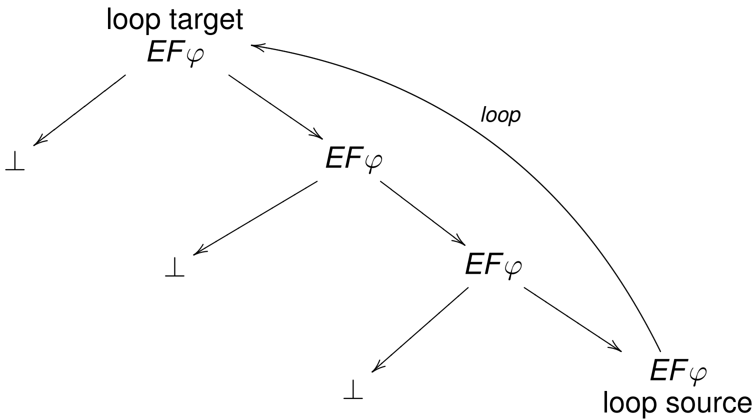
- The (EF) captures the fix-point $EF\varphi \leftrightarrow \varphi \vee EXEF\varphi$
- The (EF) captures the fix-point $AF\varphi \leftrightarrow \varphi \vee AXAF\varphi$
- The $(\vee)$ rule is standard except for computation of *uev*

Universally Branching Rules Conditions

where in the (EF), (AF) and $(\vee)$ rules :

$$
\begin{aligned}
f(x, y) &= \{i \mid (x, i) \in y\} \\
UEF &= \{(EF\psi, n) \mid f(EF\psi, uev_1) \neq \emptyset \;\&\; f(EF\psi, uev_2) \neq \emptyset \\
&\quad \& \; n = min\,(f(EF\psi, uev_1) \,\cup\, f(EF\psi, uev_2))\,\} \\
UAF &= \{(AF\psi, n) \mid f(AF\psi, uev_1) \neq \emptyset \;\&\; f(AF\psi, uev_2) \neq \emptyset \\
&\quad \& \; n = max\,(f(AF\psi, uev_1) \,\cup\, f(AF\psi, uev_2))\,\} \\
uev &:= \begin{cases} uev_1 & \text{if } uev_2 = \{(\text{false}, \_)\} \\ uev_2 & \text{if } uev_1 = \{(\text{false}, \_)\} \\ UEF \cup UAF & \text{otherwise} \end{cases}
\end{aligned}
$$

Unfolding the fix-point



Unfolding multiple fix-points

$$
\begin{tikzcd}[column sep=small]
 & & & \begin{matrix} EF\varphi; \\ EF\psi \end{matrix} \arrow[dl] \arrow[dr] & & \\
 & & \begin{matrix} \varphi; \\ EXEF\psi \end{matrix} \arrow[dl] \arrow[d] & & \begin{matrix} EXEF\varphi; \\ \psi \end{matrix} \arrow[d] \arrow[dr] & \\
 & \begin{matrix} \varphi; \\ \psi \end{matrix} & \begin{matrix} \varphi; \\ EXEF\psi \end{matrix} & & \begin{matrix} EXEF\varphi; \\ \psi \end{matrix} & \begin{matrix} EXEF\varphi; \\ EXEF\psi \end{matrix}
\end{tikzcd}
$$

Intuition for the Universally Branching Rules Conditions

The criteria for computing *uev* are as follows:

- if either branch is “closed” then keep the other uev. If both are “closed” then the $uev = uev_1 = \{(\text{false}, \_)\}$.
- if either uev_1 or uev_2 is empty because it has no “procrastinators”, then the numerator is empty because $UEF = UAF = \emptyset$, hence $UEF \cup UAF = \emptyset$.
- if both are not “closed” and both contain “procrastinators”, then $UEF \cup UAF$ constructs their “intersection” (sic!).

The crucial point is to simply ignore the notion of “open” and “fulfilled” and consider only “procrastinators”.

Existentially Branching Rules

$$(EX)\ \frac{\begin{array}{c} EX\varphi_1 \ ; \ldots ; EX\varphi_n \ ; \ EX\Gamma \ ; \ AX\Delta \ ; \ \Lambda \\ :: Fev, Br :: uev \end{array}}{\begin{array}{c} \varphi_1 \ ; \ \Delta \\ :: \emptyset, Br_1 :: uev_1 \\ \ddagger(1) \end{array} \ || \cdots || \ \begin{array}{c} \varphi_n \ ; \ \Delta \\ :: \emptyset, Br_n :: uev_n \\ \ddagger(n) \end{array} \ || \ \begin{array}{c} EX\Gamma \ ; \ AX\Delta \ ; \ \Lambda \\ :: Fev, Br :: uev_{n+1} \\ \dagger \end{array}}$$

Existentially Branching Rules Conditions

with :

$n \geq 1$

$\ddagger(i)$ is the condition $\forall j \,.\, 0 \leq j \leq len(Br) \Rightarrow \{\varphi_i\} \cup \Delta \neq Br[j].core$

$\dagger$ is the (blocking) condition $\forall \psi \in \Gamma \,.\, \exists j \geq 0$ s.t. $\{\psi\} \cup \Delta = Br[j].core$

and where for $1 \leq i \leq n$:

$$Br_i = (\{\varphi_i\} \cup \Delta, Fev).Br$$

$$l = len(Br) - 1$$

$$UEV = \bigcup_{j=1}^{n+1} uev_j$$

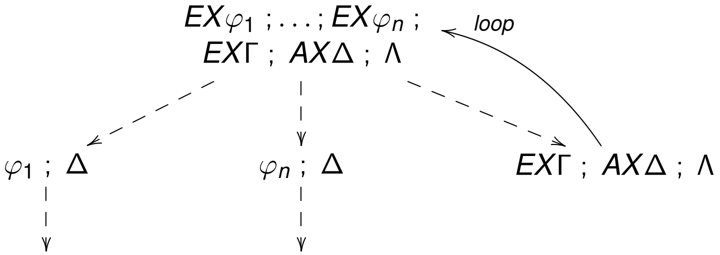
$$uev := \begin{cases} UEV & \text{if } (\text{false}, \_) \notin UEV \,\&\, \forall(\_, n) \in UEV \,.\, n \leq l \\ \{(\text{false}, l)\} & \text{otherwise} \end{cases}$$

Intuition for the Existentially Branching Rule Conditions

The criteria for computing *uev* are as follows:

- if the i^{th} branch is closed than $uev = \{(\_,\text{false})\}$.
- if any *uev* contains an iterated eventuality that loops lower then this point ($n \leq len(Br)$) then $uev = \{(\_,\text{false})\}$.
- if all eventualities loop higher than this point then *uev* is equal to the union of all *uevs*.

Intuition for the Existentially Branching Rule



Terminal Rules

$$(id)\ \frac{EX\Gamma\ ;\ AX\Delta\ ;\ \Lambda :: Fev, Br :: uev}{Stop}\ \{\neg p; p\} \subseteq \Lambda$$

where $uev := \{(\text{false}, len(Br))\}$

$$(block)\ \frac{EX\Gamma\ ;\ AX\Delta\ ;\ \Lambda :: Fev, Br :: uev}{Stop}\ \{\neg p; p\} \not\subseteq \Lambda \text{ and } \dagger$$

Block Rule Conditions

where † is the condition $\forall \psi \in \Gamma \,.\, \exists j \geq 0 \,.\, \{\psi\} \cup \Delta = Br[j].core$, and

$$
\begin{aligned}
Cores &= \{\{EF\varphi\} \cup \Delta \mid EF\varphi \in \Gamma\} \\
UAF &= \{\ (AF\varphi, i) \mid \exists\ c \in Cores, \exists\ i \text{ such that } c = Br[i].core \\
&\qquad \&\ \forall j\ .\ i \leq j \leq len(Br) \Rightarrow AF\varphi \in Br[j].core \\
&\qquad \&\ AF\varphi \notin Br[j].fev\ \} \\
UEF &= \{\ (EF\varphi, i) \mid \exists\ i \text{ such that } \{EF\varphi\} \cup \Delta = Br[i].core \\
&\qquad \&\ \forall j\ .\ i \leq j \leq len(Br) \Rightarrow EF\varphi \in Br[j].core \\
&\qquad \&\ EF\varphi \notin Br[j].fev\ \} \\
uev &:= \{(\psi, n) \in UAF \cup UEF \mid \psi \notin Fev\}
\end{aligned}
$$

Intuition for the Block Rule Conditions

A blocked iterated eventuality $(EV\varphi, n)$ is in *uev* iff

- The branch *b* from the loop target to the loop source is a procrastinator for $EV\varphi$.
- $EV\varphi$ is not fulfilled on *b* between the target and the source of the loop.

Soundness and Completeness

Proposition (Termination)

A UB-tableau for a node $\varphi :: \mathit{Fev}, \mathit{Br} :: \mathit{uev}$ always terminates.

Theorem

If $\mathcal{T}$ is an expanded tableau for $\varphi :: \mathit{Fev}, \mathit{Br} :: \mathit{uev}$, with $\mathit{uev} = \emptyset$ then φ is UB-satisfiable.

Theorem

If a formula φ is UB-satisfiable then there exists an expanded tableau for $\varphi :: \cdots :: \mathit{uev}$ where $\mathit{uev} = \emptyset$.

Outline

Conclusions

- The TWB targets both technical and non-technical users;
- It can be used to experiment with new logics, but also to built specialized TPs;
- It is based on a naive algorithms, but it can be easily extended to incorporate well known optimizations or to experiment with new ones.

Availability

- One The Web:

  `http://twb.rsise.anu.edu.au`

- Via Darcs:

  `http://twb.rsise.anu.edu.au/Repository`